Waiver Exam - ECON 897 Final Exam

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Instructions

- This is a closed-book test. No books or lecture notes are permitted.
- You have 180 minutes to complete the exam, the total score is 180.
- Read the questions carefully, and be sure to answer the questions asked.
- You can use all the results covered in all three parts.
- Please write legibly.
- Good luck!

1. Let $S \subset (M, d_1)$ and $T \subset (N, d_2)$ be such that both S and T are connected in their respective metric spaces.

   (a) **[15 points]** Let A be a *clopen* subset of $S \times T$. Assume there exists a point $(s_a, t_a) \in A \subseteq (S \times T)$ (that means A is non-empty). Show that for all $y \in T$ the point $(s_a, y) \in A$.

   (b) **[10 points]** Use part (a) to show $S \times T$ has to be connected.

2. The objective of this problem is to prove the converse of the max-min theorem. If $K \subset (M, d)$ is a set such that all continuous functions $f : K \to \mathbb{R}$ attain a maximum and a minimum, then K is compact. Complete the steps below.

   (a) **[5 points]** Assume there exists $(a_n)_{n\in\mathbb{N}}$ a sequence in K with no converging subsequence (in K). Prove that the set $A = \{a_n : n \in \mathbb{N}\}$ has infinitely many different points but no cluster points in K.

   From now on we will consider only the subsequence of the infinitely many different points of A, which we will continue to call (a_n).

   (b) **[10 points]** Use (a) to prove that for each a_n there exists a real number $r_n > 0$ such that $M_{r_n}(a_n) \cap A = \{a_n\}$, that is, there exist a positive radius such that the neighborhood around a_n with that radius contains no other point of A.

   (c) **[10 points]** Define the functions $g_n : K \to \mathbb{R}$ as $g_n(y) = d(a_n, y)$, show they are continuous.

   (d) **[5 points]** Define the functions $f_n : K \to \mathbb{R}$ as

   $$f_n(x) = \max\left\{0\,,\, 1 - \frac{1}{n} - \frac{g_n(x)}{r_n/2}\right\},$$

   where r_n is as defined in (b). Show that the functions f_n are continuous. Where do each of them attain its maximum?

   (e) **[10 points]** Show that for each $x \in K$ $f_n(x) = 0$ for all but at most one n. Then the function

   $$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

   is well defined. Why is it continuous? Show it does NOT attain a maximum.

3. **[10 points]** Suppose that f is continuous on $[a, b]$ and f'' exists on (a, b). If there is an $x_0 \in (a, b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there exists a $c \in (a, b)$ such that $f''(c) = 0$.

4. Let $f_1, f_2, \dots f_m : C \to \mathbb{R}$ be concave, let C be a non-empty convex subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}^M$ be the function $f = (f_1, f_2, \dots f_m)$. In this question we are going to prove that exactly one of the following is true.

- $\exists x \in C$ such that $f(x) >> 0$.
- $\exists p \in \mathbb{R}^M$, $p > 0$ such that $\forall x \in C$ $p \cdot f(x) \leq 0$

(a) **[5 points]**Argue that both conditions cannot hold.

(b) **[10 points]** Now suppose that the first condition fails. Show the set $H = \{y \in \mathbb{R}^m : \exists x \in C, y \leq f(x)\}$ is convex.

(c) **[10 points]** Show that there exists a non-zero vector $p \in \mathbb{R}^m$ that separates H and $\mathbb{R}_{++}$ and $p > 0$. Using this, show that the second condition holds if the first fails.

5. Let X and Y be independent, exponential random variables with parameter λ (the pdf is $f(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$.

(a) **[10 points]** Find the joint distribution of $U = X + Y$, $V = X - Y$.

(b) **[5 points]** Calculate $E(U|V)$.

6. **[10 points]** Consider the sequence of random variables $X_1, X_2, \ldots$ with pdfs

$$f_n(x) = 1 + \cos 2\pi n x \text{ on } [0, 1].$$

Show that $\{X_i\}_{i=1}^{\infty}$ converges in distribution. Find the cdf of the limit distribution.

7. **[10 points]** Consider a collection of random variables $N, X_1, X_2, \ldots$. Let $N \in \{1, 2, \ldots\}$, $X_1, X_2, \ldots$ iid and X_i and N are independent. Show that

$$E(\sum_{i=1}^{N} X_i) = E(N)E(X_1).$$

8. Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}$ satisfies $\frac{\partial f}{\partial z_i} > 0$ for all $i \in \{1, \ldots, n\}$ and $z >> 0$, and $(D^2 f)_z$ is negative definite. Let $x(p, w)$ and $z(p, w)$ denote the correspondence of solutions to:

$$\Pi(p, w) = \max_{x \in \mathbb{R}, z >> 0} px - w \cdot z;$$
$$\text{s.t. } x = f(z).$$

Assume this always has a solution.

(a) Use the first order conditions and the implicit function theorem to prove the following:

   i. **[5 points]** Find expressions for $\frac{\partial z_i}{\partial w_j}$ and $\frac{\partial z_i}{\partial p}$.

   ii. **[5 points]** Show that x is increasing in p.

   iii. **[10 points]** Show that an increase in p increases some z_i.

   iv. **[5 points]** Show that z_i is decreasing in w_i.

(b) Let $z_{-1} = (z_2, \dots z_n)$ and $w_{-1} = (w_2, \dots w_n)$. Consider the problem

$$\Pi^s(p, w, z_1) = \max_{x \in \mathbb{R}, z_{-1} >> 0} px - w_1 z_1 - w_{-1} \cdot z_{-1};$$
$$\text{s.t. } x = f(z_1, z_{-1}).$$

Let $x^s(p, w, z_1)$, and $z^s(p, w, z_1)$ be the corresponding correspondences. Show that:

i. **[5 points]** If $z_1 = z_1(p, w)$, then $x^s(p, w, z_1) = x(p, w)$ and $z^s(p, w, z_1) = z(p, w)$.

ii. **[5 points]** $\Pi^s(p, w, z_1) \leq \Pi(p, w)$.

iii. **[10 points]** Assume Π and Π^s are C^2. Use the minimization problem

$$\min_{p>0} \Pi(p, w) - \Pi^s(p, w, z_1)$$

to show that

$$\frac{\partial x^s}{\partial p}(p, w, z_1) \leq \frac{\partial x}{\partial p}(p, w)$$

for any p, w, and z_1 where $z_1 = z_1(p, w)$.